LINEARIZABILITY AND CRITICAL PERIOD BIFURCATIONS OF A GENERALIZED RICCATI SYSTEM

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Abstract. In this paper we investigate the isochronicity and linearizability problem for a cubic polynomial differential system which can be considered as a generalization of the Riccati system. Conditions for isochronicity and linearizability are found. The global structure of systems of the family with an isochronous center is determined. Furthermore, we find the order of weak center and study the problem of local bifurcation of critical periods in a neighborhood of the center.

1. Introduction

A classical problem in the qualitative theory of ordinary differential equations is to characterize the existence of centers and isochronous centers. A singular point of a planar autonomous differential system is called a *center* if all solutions sufficiently closed to it are periodic, that is, all trajectories in a small neighborhood of the singularity are ovals. If all periodic solutions inside the period annulus of the center have the same period it is said that the center is *isochronous*.

Poincaré and Lyapunov have shown that the existence of an isochronous center at the origin of a system of the form

$$\dot{x} = -y + P(x, y), \qquad \dot{y} = x + Q(x, y), \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials without constant and linear terms, is equivalent to the linearizability of the system. This equivalence has made the studies of the isochronicity problem simpler, since the linearizability problem can be extended to the complex field, where the computational methods are more efficient.

The investigation on isochronicity of oscillations started in the 17th century, when Huygens studied the cycloidal pendulum [27]. However, only in the second half of the last century the isochronicity problem began to be intensively studied. In 1964 Loud [34] found the necessary and sufficient conditions for isochronicity of system (1.1) with P and Q being quadratic homegeneous polynomials. Later on, the isochronicity problem was solved for system (1.1) when P and Q are homogeneous polynomials of degree three [40] (see also [29]) and degree five [41]. However in the case of the linear center perturbed by homogeneous polynomials of degree four the problem is still unsolved, although some partial results were obtained [7, 23]. The reason is that linearizability quantities (which are polynomials in the parameters of system (1.1) defined at the beginning of Section 2) have more complicate expressions in the case of homogeneous perturbations of degree four, than in the case of homogeneous perturbations of degree five. There are also many works devoted to the investigation of

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particular families of some other polynomial systems, see e.g. [1, 6, 9, 14, 35, 43] and references therein. Many works also deal with investigation of isochronicity of Hamiltonian systems, see e.g. [12, 15, 24, 28, 31] and references given there.

The problem of critical period bifurcations is tightly related to the isochronicity problem. In a neighborhood of a center the so-called *period function* $T(r)$ gives the least period of the periodic solution passing through the point with coordinates $(x, y) = (r, 0)$ inside the period annulus of the center. For a center that is not isochronous any value $r > 0$ for which $T'(r) = 0$ is called a critical period. The problem of critical period bifurcations is aimed on estimating of the number of critical periods that can arise near the center under small perturbations. In 1989, Chicone and Jacobs [11] introduced for the first time the theory of local bifurcations of critical periods and solved the problem for the quadratic system. Local bifurcations of critical periods have been investigated for cubic systems with homogeneous nonlinearities [45], the reduced Kukles system [46], the Kolmogorov system [10], the Z_2-equivariant systems [8] and some other families (see e.g. [16, 22, 49] and references therein). In [20] a general approach to studying bifurcations of critical periods based on a complexification of the system was described, and some upper bounds on the number of critical periods of several cubic systems were obtained.

In this paper we are interested in the family of Riccati systems. The classic *Riccati system* is written in the form

$$\dot{x} = 1, \qquad \dot{y} = g_2(x)y^2 + g_1(x)y + g_0(x), \tag{1.2}$$

where each $g_j(x)$ is a $\mathcal{C}^1$ function with respect to x and $g_2(x)g_0(x) \not\equiv 0$. System (1.2) becomes a special case of Berouilli system if $g_0(x) \equiv 0$, and it obviously is a linear differential system if $g_2(x) \equiv 0$.

The Riccati equation has been invstigated by many authors, see for example [32, 33] and references therein. They are important since they can be used to solve second-order ordinary differential equations and can be applied in studying the third-order Schwarzian differential [37]. It also has many applications in both physics and mathematics. For instance, renormalization group equations for running coupling constants in quantum field theories [5], nonlinear physics [36], Newton's laws of motion [39], thermodynamics [44] and variational calculus [50].

Recently Llibre and Valls [32, 33] investigated the planar differential system

$$\dot{x} = f(y), \qquad \dot{y} = g_2(x)y^2 + g_1(x)y + g_0(x),$$

which is called the *generalized Riccati system*, since it becomes the classic Riccati system when $f(y) \equiv 1$. In this paper we study a subfamily of the generalized Riccati system, cubic systems of the form

$$\begin{aligned} \dot{x} &= -y + a_{02}y^2 + a_{03}y^3, \\ \dot{y} &= (b_{02} + b_{12}x)y^2 + (b_{11}x + b_{21}x^2)y + (x + b_{20}x^2 + b_{30}x^3) \\ &= x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2, \end{aligned} \tag{1.3}$$

where x, y are unknown real functions and a_{ij}, b_{ij} are real parameters. Note that system (1.3) is the so-called reduced Kukles system when $a_{02} = a_{03} = 0$.

The aims of our study are to obtain conditions on parameters a_{ij} and b_{ij} for the linearizability of system (1.3), to study the global structures of trajectories when the system has an isochronous center, and to investigate the local bifurcations of critical periods at the origin. In Section 2 we present our main result on linearizability, Theorem 2.1, which gives conditions for the linearizability of system (1.3). We also describe an approach for deriving such conditions which is based on making use of modular computations which are performed in the systems of computer algebra SINGULAR [17] and MATHEMATICA [48]. The approach can be applied to investigate many problems involving solving systems of algebraic polynomials. In Section 3 we study the global dynamics of system (1.3) when the origin is an isochronous center. The last section is devoted to the investigation of local bifurcations of critical periods in a neighborhood of the center.

2. LINEARIZABILITY OF SYSTEM (1.3)

We first briefly remind an approach for studying the isochronicity and linearizability problems for polynomial differential systems of the form

$$\dot{x} = -y + \sum_{p+q=2}^{n} a_{p,q} x^p y^q, \quad \dot{y} = x + \sum_{p+q=2}^{n} b_{p,q} x^p y^q, \tag{2.1}$$

where x, y and $a_{p,q}, b_{p,q}$ are in $\mathbb{R}$.

System (2.1) is *linearizable* if there is an analytic change of coordinates

$$x_1 = x + \sum_{m+n\geq 2} c_{m,n} x^m y^n, \qquad y_1 = y + \sum_{m+n\geq 2} d_{m,n} x^m y^n, \tag{2.2}$$

which reduces (2.1) to the canonical linear system $\dot{x}_1 = -y_1$, $\dot{y}_1 = x_1$.

Obstacles for existence of a transformation (2.2) are some polynomials in parameters of system (2.1) called the *linearizability quantities* and denoted by i_k, j_k $(k = 1, 2, ...)$.

Differentiating with respect to t both sides of each equation of (2.2) we obtain

$$\begin{aligned} \dot{x}_1 =& \dot{x} + \left(\sum_{m+n\geq 2} m c_{m,n} x^{m-1} y^n\right) \dot{x} + \left(\sum_{m+n\geq 2} n c_{m,n} x^m y^{n-1}\right) \dot{y}, \\ \dot{y}_1 =& \dot{y} + \left(\sum_{m+n\geq 2} m d_{m,n} x^{m-1} y^n\right) \dot{x} + \left(\sum_{m+n\geq 2} n d_{m,n} x^m y^{n-1}\right) \dot{y}. \end{aligned} \tag{2.3}$$

Substituting in the above equations the expressions from (2.2) and (2.1), one computes the linearizability quantities i_k, j_k step-by-step (see e.g. [21] for more details).

From (2.3) it is easy to see that the linearizability quantities i_k, j_k are polynomials in parameters $a_{p,q}, b_{p,q}$ of system (2.1). We denote by (a, b) the s-tuple (s is the number of parameters $a_{p,q}, b_{p,q}$ in system (2.1)) of parameters of (2.1), so $(a, b) = (a_{2,0}, a_{1,1}, \ldots, b_{0,n})$, and by $\mathbb{R}[a, b]$ and $\mathbb{C}[a, b]$ the rings of polynomials in $a_{p,q}, b_{p,q}$ with real and complex coefficients, respectively.

Thus, the simultaneous vanishing of all linearizability quantities i_k, j_k provides conditions which characterize when a system of the form (2.1) is linearizable. The ideal defined by the linearizability quantities, $\mathcal{L} = \langle i_1, j_1, i_2, j_2, ...\rangle \subset \mathbb{R}[a, b]$, is called the *linearizability ideal* and its affine variety, $V_{\mathcal{L}} = \mathbf{V}(\mathcal{L})$ is called the *linearizability variety*.

In order to find a linearizing change of coordinates explicitly one can look for Darboux linearization. To construct a Darboux linearization for system (2.1) it is convenient to complexify the system using the substitution

$$z = x + iy, \qquad w = x - iy. \tag{2.4}$$

Then, after a time rescaling by i we obtain from (2.1) a system of the form

$$\dot{z} = z + X(z, w), \qquad \dot{w} = -w - Y(z, w). \tag{2.5}$$

System (2.1) is linearizable if and only if system (2.5) is linearizable.

A *Darboux factor* of system (2.5) is a polynomial $f(z, w)$ satisfying

$$\frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial w}\dot{w} = Kf,$$

where polynomial $K(z, w)$ is called the *cofactor of* f. A *Darboux linearization* of system (2.5) is an analytic change of coordinates $z_1 = Z_1(z, w)$, $w_1 = W_1(z, w)$, such that

$$Z_1(z, w) = \prod_{j=0}^{m} f_j^{\alpha_j}(z, w) = z + \tilde{Z}_1(z, w),$$

$$W_1(z, w) = \prod_{j=0}^{n} g_j^{\beta_j}(z, w) = w + \tilde{W}_1(z, w),$$

which linearizes (2.5), where $f_j, g_j \in \mathbb{C}[z, w]$, $\alpha_j, \beta_j \in \mathbb{C}$, and $\tilde{Z}_1$ and $\tilde{W}_1$ have neither constant terms nor linear terms.

It is easy to see that system (2.5) is Darboux linearizable if there exist $s + 1 \geq 1$ Darboux factors $f_0, ..., f_s$ with corresponding cofactors $K_0, ..., K_s$, and $t + 1 \geq 1$ Darboux factors $g_0, ..., g_t$ with corresponding cofactors $L_0, ..., L_t$ with the following properties:

(i) $f_0(z, w) = z + \cdots$ but $f_j(0, 0) = 1$ for $j \geq 1$;
(ii) $g_0(z, w) = w + \cdots$ but $g_j(0, 0) = 1$ for $j \geq 1$; and
(iii) there are $s + t$ constants $\alpha_1, ..., \alpha_s, \beta_1, ..., \beta_t \in \mathbb{C}$ such that

$$K_0 + \alpha_1 K_1 + \cdots + \alpha_s K_s = 1 \quad \text{and} \quad L_0 + \beta_1 L_1 + \cdots + \beta_t L_t = -1. \tag{2.6}$$

The Darboux linearization is then given by the transformations

$$z_1 = H_1(z, w) = f_0 f_1^{\alpha_1} \cdots f_s^{\alpha_s}, \qquad y_1 = H_2(z, w) = g_0 g_1^{\beta_1} \cdots g_t^{\beta_t}.$$

The readers can consult [13, 35, 43] for more details.

Before passing to the results of our paper we remind some fact about solutions of systems of nonlinear polynomial equations which we will need for our study.

Denote by $k[x_1, \ldots, x_n]$ the ring of polynomials with coefficients in a field k and consider a system of polynomials of $k[x_1, \ldots, x_n]$:

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= 0. \end{aligned} \tag{2.7}$$

We recall that the ideal I in $k[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_m$, denoted by $I = \langle f_1, \ldots, f_m \rangle$, is the set of all polynomials of $k[x_1, \ldots, x_n]$ expressed in the form $f_1 h_1 + f_2 h_2 + \cdots + f_m h_m$, where $h_1, h_2, \ldots, h_m$ are polynomials of $k[x_1, \ldots, x_n]$. The variety of the ideal $I = \langle f_1, \ldots, f_m \rangle \subset k[x_1, \ldots, x_n]$ in k^n, denoted by $\mathbf{V}(I)$, is the zero set of all polynomials of I,

$$\mathbf{V}(I) = \{A = (a_1, \ldots, a_n) \in k^n | f(A) = 0 \quad \text{for all } f \in I\}.$$

The situation when the variety of a polynomial ideal consists of a finite number of points arises very rarely. In a generic case, the variety consists of infinitely many points, so generally speaking,"to solve" system (2.7) means to find a decomposition of the variety of the ideal into irreducible components. More precisely, an affine variety $V \subset k^n$ is *irreducible* if, whenever $V = V_1 \cup V_2$ for affine varieties V_1 and V_2, then either $V_1 = V$ or $V_2 = V$. Let I be an ideal and $V = \mathbf{V}(I)$ its variety. Then V can be represented as a union of irreducible components, $V = V_1 \cup \cdots \cup V_m$, where each V_i is irreducible. The radical of I denoted by $\sqrt{I}$ is the set of all polynomials f of $k[x_1, \ldots, x_n]$ such that for some non-negative integer p f^p is in I. Clearly, I and $\sqrt{I}$ have the same varieties. It is known that $\sqrt{I}$ can be expressed as an intersection of prime ideals, $\sqrt{I} = \cap_{j=1}^{s} Q_j$. Prime ideals Q_i are called *the minimal associate primes of* I. Let V_i $(i = 1, \ldots, s)$ be the variety of Q_i. Since the variety of an intersection of some ideals is equal to the union of the varieties of the ideals, we have that $\mathbf{V}(I) = \mathbf{V}(\sqrt{I}) = \cap_{j=1}^{s} V_j$. For example, if $I = \langle x^2y^3, xz^5 \rangle$, then $\sqrt{I} = \langle xy, xz \rangle = \langle x \rangle \cap \langle y, z \rangle$, that is, the variety of I is the union of two irreducible components: the plane $x = 0$ and the line $y = z = 0$. In the computer algebra system SINGULAR [17] one can compute the minimal associate primes of a given polynomial ideal and, thus, the irreducible decomposition of its variety using the routine `minAssGTZ`.

Proceeding now to the results of our paper we first state the following theorem on the linearizability of system (1.3).

Theorem 2.1. *System* (1.3) *is linearizable at the origin if one of the following conditions holds:*

(1) $b_{12} = a_{02} = b_{30} = b_{21} = a_{03} = b_{02} + b_{20} = b_{11}^2 + 4b_{20}^2 = 0,$
(2) $b_{12} = a_{02} = b_{20} = b_{02} = b_{21} = a_{03} = 9b_{30} - b_{11}^2 = 0,$
(3) $b_{12} = a_{02} = b_{11} = b_{20} = b_{30} = b_{21} = 9a_{03} + 4b_{02}^2 = 0,$
(4) $b_{12} = b_{30} = b_{21} = a_{03} = 2b_{02} + 5b_{20} = 10a_{02} - 3b_{11} = 4b_{11}^2 + 25b_{20}^2 = 0.$

Proof. Using the computer algebra system MATHEMATICA and the standard procedure mentioned above for system (1.3) we have computed the first eight pairs of the linearizability quantities $i_1, j_1, ..., i_8, j_8$. Their expressions are very large, so we only present the first two pairs in the Appendix. The reader can easily compute the other quantities using any available computer algebra system [1].

The next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_8) = \mathbf{V}(\langle i_1, j_1, ..., i_8, j_8 \rangle)$.

Performing the computations by the routine `minAssGTZ` [18] of SINGULAR [17] over the field of characteristic 32452843 we obtain that $\mathbf{V}(\mathcal{L}_9)$ is equal to the union of the varieties of four ideals. After lifting these four ideals to the ring of polynomials with rational coefficients

[1] One can download linearizability quantities i_1, j_1, …, i_8, j_8 and the SINGULAR code to perform the decomposition of the variety from http://teacher.shnu.edu.cn/_upload/article/files/79/14/f36e87e342b8b0d6977e6debdeb3/3b818cf4-a6f7-4f07-8669-f4b78e48f733.txt.

using the rational reconstruction algorithm of [47] we obtain the ideals

$$\begin{aligned}
J_1 =& \langle b_{12}, a_{02}, b_{30}, b_{21}, a_{03}, b_{02}+b_{20}, b_{11}^2+4b_{20}^2\rangle,\\
J_2 =& \langle b_{12}, a_{02}, b_{20}, b_{02}, b_{21}, a_{03}, 9b_{30}-b_{11}^2\rangle,\\
J_3 =& \langle b_{12}, a_{02}, b_{11}, b_{20}, b_{30}, b_{21}, 9a_{03}+4b_{02}^2\rangle,\\
J_4 =& \langle b_{12}, b_{30}, b_{21}, a_{03}, 2b_{02}+5b_{20}, 10a_{02}-3b_{11}, 4b_{11}^2+25b_{20}^2\rangle.
\end{aligned}$$

The varieties of J_1, J_2, J_3 and J_4 provide conditions (1), (2), (3) and (4) of the theorem, respectively.

To check the correctness of the obtained conditions we use the procedure described in [42]. First, we computed the ideal $J = J_1 \cap J_2 \cap J_3 \cap J_4$, which defines the union of all four sets given in the statement of the theorem. Then we check that $\mathbf{V}(J) = \mathbf{V}(\mathcal{L}_8)$. According to the Radical Membership Test, to verify the inclusion $\mathbf{V}(J) \supset \mathbf{V}(\mathcal{L}_8)$ it is sufficient to check that the Groebner bases of all ideals $\langle J, 1-wi_k\rangle$, $\langle J, 1-wj_k\rangle$ (where $k=1,\dots,9$ and w is a new variable) computed over $\mathbb{Q}$ are $\{1\}$. The computations show that this is the case. To check the opposite inclusion, $\mathbf{V}(J) \subset \mathbf{V}(\mathcal{L}_8)$, it is sufficient to check that Groebner bases of the ideals $\langle \mathcal{L}_8, 1-wf_i\rangle$ (where the polynomials f_i's are the polynomials of a basis of J) computed over $\mathbb{Q}$ are equal to $\{1\}$. Unfortunately, we were not able to perform these computations over $\mathbb{Q}$ however we have checked that all the bases are $\{1\}$ over few fields of finite characteristic. It yields that the list of conditions in Theorem 2.1 is the complete list of linearizability conditions for system (1.3) with high probability [3].

We now prove that under each of conditions (1)–(4) of the theorem the system is linearizable.

Condition (1). In this case $b_{11} = \pm 2b_{20}i$. We consider only the case $b_{11} = 2b_{20}i$, since when $b_{11} = -2b_{20}i$ the proof is analogous. After the change of variables (2.4) system (1.3) becomes

$$\begin{aligned}
\dot z =& z + b_{20}z^2,\\
\dot w =& -w - b_{20}z^2,
\end{aligned} \tag{2.8}$$

which is a quadratic system. By Theorem 3.1 of [13] and Theorem 4.5.1 of [43] system (2.8) is Darboux linearizable and, therefore, system (1.3) is linearizable if condition (1) holds.

Condition (2). After substitution (2.4) system (1.3) becomes

$$\begin{aligned}
\dot z =& z + \frac{1}{72}(-18ib_{11}z^2 + 18ib_{11}w^2 + b_{11}^2z^3 + 3b_{11}^2z^2w + 3b_{11}^2zw^2 + b_{11}^2w^3),\\
\dot w =& -w + \frac{1}{72}(18ib_{11}z^2 - 18ib_{11}w^2 - b_{11}^2z^3 - 3b_{11}^2z^2w - 3b_{11}^2zw^2 - b_{11}^2w^3).
\end{aligned} \tag{2.9}$$

It has the Darboux factors

$$\begin{aligned}
l_1 =& z + \frac{ib_{11}}{12}z^2 + \frac{ib_{11}}{6}zw + \frac{ib_{11}}{12}w^2,\\
l_2 =& w - \frac{ib_{11}}{12}z^2 - \frac{ib_{11}}{6}zw - \frac{ib_{11}}{12}w^2,\\
l_3 =& 1 - \frac{ib_{11}}{6}z + \frac{b_{11}^2}{36}z^2 + \frac{ib_{11}}{6}w + \frac{b_{11}^2}{18}zw + \frac{b_{11}^2}{36}w^2,\\
l_4 =& 1 - \frac{ib_{11}}{3}z + \frac{b_{11}^2}{36}z^2 + \frac{ib_{11}}{3}w + \frac{b_{11}^2}{18}zw + \frac{b_{11}^2}{36}w^2
\end{aligned}$$

with the respective cofactors

$$k_1 = 1 - \frac{ib_{11}}{6}z - \frac{ib_{11}}{6}w, \quad k_2 = -1 - \frac{ib_{11}}{6}z - \frac{ib_{11}}{6}w,$$
$$k_3 = -\frac{ib_{11}}{6}z - \frac{ib_{11}}{6}w, \qquad k_4 = -\frac{ib_{11}}{3}z - \frac{ib_{11}}{3}w.$$

It is easy to verify that (2.6) is satisfied with $\alpha_1 = 1,\ \alpha_2 = -1,\ \beta_1 = 1$ and $\beta_2 = -1$. Hence the Darboux linearization for system (2.9) is given by the analytic change of coordinates

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2}.$$

Thus, system (2.9) is linearizable and therefore the corresponding system (1.3) is linearizable as well.

Condition (3). In this case after substitution (2.4) the corresponding system (1.3) is changed to

$$\begin{aligned} \dot{z} =& z + \frac{1}{36}(-9b_{02}z^2 + 18b_{02}zw - 9b_{02}w^2 - 2b_{02}^2z^3 + 6b_{02}^2z^2w - 6b_{02}^2zw^2 + 2b_{02}^2w^3), \\ \dot{w} =& -w + \frac{1}{36}(9b_{02}z^2 - 18b_{02}zw + 9b_{02}w^2 - 2b_{02}^2z^3 + 6b_{02}^2z^2w - 6b_{02}^2zw^2 + 2b_{02}^2w^3). \end{aligned} \tag{2.10}$$

System (2.10) has the Darboux factors

$$\begin{aligned} l_1 =& z - \frac{b_{02}}{12}z^2 + \frac{b_{02}}{6}zw - \frac{b_{02}}{12}w^2, \\ l_2 =& w - \frac{b_{02}}{12}z^2 + \frac{b_{02}}{6}zw - \frac{b_{02}}{12}w^2, \\ l_3 =& 1 - \frac{2b_{02}}{3}z + \frac{2b_{02}^2}{9}z^2 - \frac{2b_{02}}{3}w - \frac{4b_{02}^2}{9}zw + \frac{2b_{02}^2}{9}w^2, \\ l_4 =& 1 - \frac{b_{02}}{3}z + \frac{b_{02}^2}{18}z^2 - \frac{b_{02}}{3}w - \frac{b_{02}^2}{9}zw + \frac{b_{02}^2}{18}w^2, \end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where $\alpha_1 = 1,\ \alpha_2 = -3,\ \beta_1 = 1$ and $\beta_2 = -3$.

Condition (4). For this condition it is easy to see that $b_{11} = \pm 5b_{20}i/2$. We consider only the case $b_{11} = 5b_{20}i/2$, since when $b_{11} = -5b_{20}i/2$ the proof is analogous. After transformation (2.4) system (1.3) becomes

$$\begin{aligned} \dot{z} =& z + \frac{1}{16}(21b_{20}z^2 - 6b_{20}zw + b_{20}w^2), \\ \dot{w} =& -w + \frac{1}{16}(-27b_{20}z^2 + 18b_{20}zw - 7b_{20}w^2). \end{aligned} \tag{2.11}$$

System (2.11) has the Darboux factors

$$\begin{aligned}
l_1 =& z + \frac{1}{16}3b_{20}z^2 + \frac{1}{8}b_{20}zw + \frac{1}{48}b_{20}w^2,\\
l_2 =& w + \frac{1}{16}9b_{20}z^2 + \frac{3b_{20}}{8}zw + \frac{b_{20}}{16}w^2,\\
l_3 =& 1 + 3b_{20}z + \frac{27b_{20}^2}{8}z^2 + b_{20}w - \frac{3b_{20}^2}{4}zw + \frac{3b_{20}^2}{8}w^2,\\
l_4 =& 1 + \frac{3b_{20}}{2}z + \frac{9b_{20}^2}{16}z^2 + \frac{b_{20}}{2}w + \frac{3b_{20}^2}{8}zw + \frac{b_{20}^2}{16}w^2,
\end{aligned}$$

yielding the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where $\alpha_1 = 1,\ \alpha_2 = -3,\ \beta_1 = 1$ and $\beta_2 = -3$. □

3. Global dynamics of system (1.3) having an isochronous center

Global phase portrait of a planar autonomous system is usually plotted on the Poincaré disc, which is obtained using the Poincaré compactification. We remind the procedure briefly, for more details see for instance [2, 19].

Consider the planar vector field

$$\mathcal{X} = \tilde{P}(x,y)\frac{\partial}{\partial x} + \tilde{Q}(x,y)\frac{\partial}{\partial y},$$

where $\tilde{P}(x,y)$ and $\tilde{Q}(x,y)$ are polynomials of degree n. Let $\mathbb{S}^2 = \{y = (y_1,y_2,y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, $\mathbb{S}^1$ be the equator of $\mathbb{S}^2$ and $p(\mathcal{X})$ be the *Poincaré compactification* of $\mathcal{X}$ on $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and once we know the behaviour of $p(\mathcal{X})$ near $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ in a neighbourhood of the infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1,y_2,y_3) \mapsto (y_1,y_2)$ is called the *Poincaré disc*, and its boundary is $\mathbb{S}^1$.

Because $\mathbb{S}^2$ is a differentiable manifold, we consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for computing the expression of $p(\mathcal{X})$ where $i = 1,2,3$. The diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1,2,3$ are the inverses of the central projections from the planes tangent at the points $(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1)$, and $(0,0,-1)$ respectively. We denote by (u,v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1,2,3$.

The expression for $p(\mathcal{X})$ in the local chart (U_1,F_1) is given by

$$\dot{u} = v^n\left[-u\tilde{P}\left(\frac{1}{v},\frac{u}{v}\right) + \tilde{Q}\left(\frac{1}{v},\frac{u}{v}\right)\right], \quad \dot{v} = -v^{n+1}\tilde{P}\left(\frac{1}{v},\frac{u}{v}\right),$$

for (U_2,F_2) is

$$\dot{u} = v^n\left[\tilde{P}\left(\frac{u}{v},\frac{1}{v}\right) - u\tilde{Q}\left(\frac{u}{v},\frac{1}{v}\right)\right], \quad \dot{v} = -v^{n+1}\tilde{Q}\left(\frac{u}{v},\frac{1}{v}\right),$$

and for (U_3,F_3) is

$$\dot{u} = \tilde{P}(u,v), \quad \dot{v} = \tilde{Q}(u,v).$$

The expressions for V_i's are the same as that for U_i's but multiplied by the factor $(-1)^{n-1}$. In these coordinates $v = 0$ always denotes the points of $\mathbb{S}^1$. When we study the infinite singular points on the charts $U_2 \cup V_2$, we only need to verify if the origin of these charts are singular points.

It is said that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$, preserving or not the sense of all orbits.

In this section, we study the global structures of system (1.3) in Poincaré discs for the case when it has an isochronous center listed in Theorem 2.1.

Theorem 3.1. *The global phase portrait of system* (1.3) *possessing an isochronous center listed in Theorem 2.1 is topologically equivalent to one of phase portraits in Fig. 1. More precisely, there exists only one equilibrium of system* (1.3) *in the plane, which is an isochronous center at the origin. The neighborhood of equilibrium at infinity consists of one elliptic sector and three hyperbolic sectors (or two hyperbolic sectors and two parabolic sectors) under conditions* (2) *and* $b_{11} \neq 0$ *(or under conditions* (3) *and* $b_{02} \neq 0$*); otherwise, the isochronous center is global.*

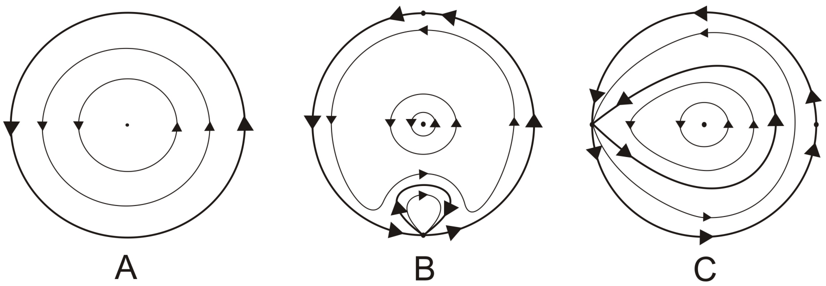

FIGURE 1. Global phase portraits of system (1.3) possessing an isochronous center listed in Theorem 2.1.

Proof. From Theorem 2.1 we have that under conditions (1)–(4) system (1.3) is linearizable. Under conditions (1) and (4) real systems (1.3) becomes the linear system $\dot{x} = -y,\ \dot{y} = x$ and its phase portrait is presented in Figure 1.A.

Under conditions (2) and (3) system (1.3) becomes

$$\dot{x} = -y, \qquad \dot{y} = x + b_{11}xy + \frac{b_{11}^2}{9}x^3, \tag{3.1}$$

and

$$\dot{x} = -y - \frac{4}{9}b_{02}^2 y^3, \qquad \dot{y} = x + b_{02}y^2, \tag{3.2}$$

respectively.

Note that if $b_{11} = 0$ in (3.1) and $b_{02} = 0$ in (3.2), then both systems are the canonic linear systems and have a global center shown in Figure 1.A. Thus, we consider the cases when

$b_{11} \neq 0$ and $b_{02} \neq 0$. In both cases by a linear change of coordinates we can reduce systems (3.1) and (3.2) to systems

$$\dot{x} = -y, \qquad \dot{y} = x + xy + \frac{x^3}{9}, \tag{3.3}$$

and

$$\dot{x} = -y - \frac{4}{9}y^3, \qquad \dot{y} = x + y^2, \tag{3.4}$$

respectively.

System (3.3) has only the isochronous center at $(0,0)$ as a finite singular point. Now we analyze its singular points at infinity. In the local chart U_1 system (3.3) becomes

$$\dot{u} = \frac{1}{9}(1 + 9uv + 9v^2 + 9u^2v^2), \qquad \dot{v} = uv^3.$$

This system has no real singular points. So the unique possible infinite singular point is the origin of the local chart U_2. In the local chart U_2 system (3.3) becomes

$$\dot{u} = \frac{1}{9}(-u^4 - 9u^2v - 9v^2 - 9u^2v^2), \qquad \dot{v} = -\frac{1}{9}uv(u^2 + 9v + 9v^2). \tag{3.5}$$

It is clear that $(0,0)$ is a singular point of (3.5) and the linear part of (3.5) at $(0,0)$ is the null matrix, i.e, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Applying the directional blow-up in the v-axis twice we obtain that the behaviour of the orbits close to the origin of U_2 is as in Figure 2. Therefore, the global phase portrait of system (3.3) is topologically equivalent to the one in Figure 1.B.

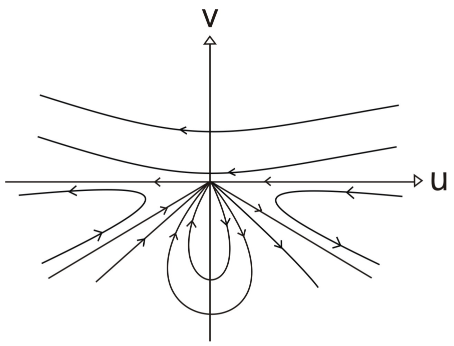

FIGURE 2. Behaviour of the orbits close to the origin of system (3.5).

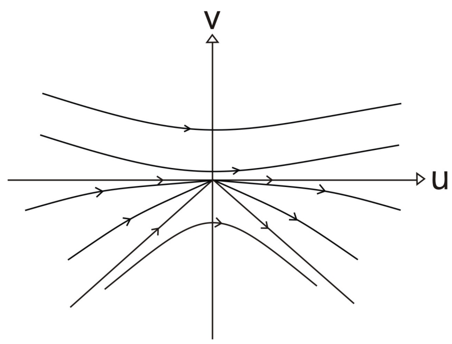

FIGURE 3. Behaviour of the orbits close to the origin of system (3.6).

Now we study system (3.4). This system has only the isochronous center at $(0,0)$ as a finite singular point. For the infinite singular points, in the local chart U_1 system (3.4) becomes

$$\dot{u} = \frac{1}{9}(4u^4 + 9u^2v + 9v^2 + 9u^2v^2), \qquad \dot{v} = \frac{1}{9}uv(4u^2 + 9v^2). \tag{3.6}$$

This system has only $(0,0)$ as a singular point, and the linear part of (3.6) at $(0,0)$ is the null matrix. Applying the directional blow-up in the v-axis twice we obtain that the behaviour of the orbits close to the origin of U_1 is as showing in Figure 3.

In the local chart U_2 system (3.4) becomes

$$\dot{u} = \frac{1}{9}(-4 - 9uv - 9v^2 - 9u^2v^2), \qquad \dot{v} = -v^2(1 + uv). \tag{3.7}$$

As it is mentioned above, we need to study only the origin of this chart, but $(0,0)$ is not a singular point for system (3.7). Thus, the global phase portrait of system (3.4) is topologically equivalent to the portrait in Figure 1.C. □

4. Weak center and local bifurcation of critical periods

Let $\alpha = (a_{20}, a_{11}, ..., b_{20}, b_{11}, ...)$ be the string of parameters of real system (2.1) with a center at the origin. Changing the system to the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and eliminating t, we obtain

$$\frac{dr}{d\theta} = r\frac{x\dot{x} + y\dot{y}}{x\dot{y} - y\dot{x}} = \frac{rH(r,\theta,\alpha)}{1 + G(r,\theta,\alpha)}, \tag{4.1}$$

where $H(r,\theta,\alpha)$ and $G(r,\theta,\alpha)$ are polynomials of $r, \alpha, \cos\theta$ and $\sin\theta$. The solution $r = r(\theta,\alpha)$ of equation (4.1) satisfying the initial condition $r(0,\alpha) = r_0 > 0$ may be locally represented as a convergent power series in r_0,

$$r(\theta,\alpha) = \sum_{k=1}^{\infty} v_k(\theta,\alpha) r_0^k. \tag{4.2}$$

Substituting (4.2) into (4.1), one can find coefficients $v_k(\theta,\alpha)$ $(k > 1)$ by successive integration.

Assuming that Γ_{r_0} is the closed trajectory through $(r_0, 0)$, we can compute the period function as

$$T(r_0,\alpha) = \oint_{\Gamma_{r_0}} dt = \int_0^{2\pi} \frac{d\theta}{1 + G(r,\theta,\alpha)} = \sum_{k=0}^{\infty} p_k(\alpha) r_0^k.$$

The period function is even and has the Taylor series expansion

$$T(r_0,\alpha) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\alpha) r_0^{2k}, \tag{4.3}$$

where $r_0 < \delta$ and coefficients p_{2k}'s are polynomials in parameters of system (2.1) (see e.g. [1, 11, 35, 43]).

If $p_2 = ... = p_{2k} = 0$ and $p_{2k+2} \neq 0$, then the origin of system (2.1) is a *weak center* of order k. If $p_{2k} = 0$ for each $k \geq 1$, then the origin is an *isochronous center*. For a center which is not isochronous, a *local critical period* is any value $\tilde{r}_0 < \delta$ for which $T'(\tilde{r}_0) = 0$.

By classical results of local critical period bifurcations [11], at most k local critical periods can bifurcate from the period function related to a weak center of order k. In order to prove that there are perturbations with exactly k local critical periods, we remind Theorem 2 of [49] as follows.

Theorem 4.1. *Assume that the period constants p_{2j} $(j = 1, 2, ..., k)$ of system (2.1) depend on k independent parameters $a_1, a_2, ..., a_k$. Suppose that there exists $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_k)$ such that*

$$p_{2j}(\tilde{a}) = p_{2j}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_k) = 0, \quad j = 1, 2, ..., k,$$
$$p_{2k+2}(\tilde{a}) \neq 0$$

and

$$\det\left(\frac{\partial(p_2, p_4, ..., p_{2k})}{\partial(a_1, a_2, ..., a_k)}(\tilde{a})\right) \neq 0,$$

then k critical periods bifurcate from the center at the origin of system (2.1) after small appropriate perturbations.

Remark. The proof that k critical periods can bifurcate after perturbations of system (2.1) corresponding to parameters $\tilde{a}$ is derived using the Implicit Function Theorem, and the proof that the bound k is sharp can be derived either using the Mean Values Theorem [26] or Rolle's Theorem [4]. In practice k critical periods can be obtained choosing perturbations such that for some system a^* close to

$$|p_2(a^*)| \ll |p_4(a^*)| \ll \cdots \ll |p_{2k}(a^*)| \ll |p_{2k+2}(a^*)|$$

and the signs in the sequence $p_2(a^*), p_4(a^*), \ldots p_{2k}(a^*), p_{2k+2}(a^*)$ alternate (see e.g. [25, 30, 43] for more details).

Because bifurcations of critical periods are bifurcations from centers, to study them for system (1.3) we need to know the center variety of the system. Due to computational difficulties the center variety of system (1.3) has been found only in the case when $a_{03} = 0$ [51]. So, from now on we assume that in system (1.3) $a_{03} = 0$ and consider the system

$$\begin{aligned} \dot{x} &= -y + a_{02}y^2, \\ \dot{y} &= x + (b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2y + b_{12}xy^2). \end{aligned} \tag{4.4}$$

The centers of system (4.4) are identified in the following theorem.

Theorem 4.2 ([51]). *System (4.4) has a center at the origin if the 7-tuple of its parameters belongs to the variety of one of the following prime ideals:*

(1) $I_1 = \langle b_{21}, b_{20}, b_{02}\rangle$,
(2) $I_2 = \langle b_{30}, b_{12}, b_{02}, b_{11}b_{20} - b_{21}\rangle$,
(3) $I_3 = \langle b_{30}, b_{21}, b_{12}, -2b_{02}b_{11}^2 + 4b_{02}^2b_{20} - b_{11}^2b_{20}, 2a_{02}b_{11} + b_{11}^2 - 4b_{02}b_{20}, 2a_{02}b_{02} - b_{02}b_{11} - b_{11}b_{20}, 4a_{02}^2 - b_{11}^2 - 4b_{20}^2\rangle$,
(4) $I_4 = \langle b_{21}, b_{11}, a_{02}\rangle$,
(5) $I_5 = \langle a_{02}, b_{02}b_{21} + b_{11}b_{30}, 2b_{02}b_{12} + b_{12}b_{20} + b_{02}b_{30}, b_{02}b_{11} + b_{11}b_{20} - b_{21}, b_{02}^2 + b_{02}b_{20} + b_{30}, b_{12}b_{20}b_{21} - 2b_{11}b_{12}b_{30} - b_{11}b_{30}^2, b_{11}b_{20}b_{21} - b_{21}^2 - b_{11}^2b_{30}, b_{12}b_{20}^2 - 4b_{12}b_{30} - b_{02}b_{20}b_{30} - 2b_{30}^2, b_{11}b_{12}b_{20} - 2b_{12}b_{21} + b_{11}b_{20}b_{30} - b_{21}b_{30}, -(b_{12}b_{21}^2) + b_{11}^2b_{12}b_{30} + b_{11}^2b_{30}^2\rangle$,
(6) $I_6 = \langle b_{21}, b_{12}, b_{11}, b_{02}\rangle$,
(7) $I_7 = \langle b_{21}, b_{12}, b_{30}, 3b_{02} + 5b_{20}, 5a_{02} - b_{11}, 6b_{11}^2 + 25b_{20}^2\rangle$.

Remark. Like in the proof of our Theorem 2.1 modular computations were used in order to determine centers of system (4.4), so it can happen that the list of centers of the system given in Theorem 4.2 is incomplete. For this reason it stands in the theorem "if" but not "if and only if".

We consider the local bifurcations of critical periods for system (4.4) when all parameters are real. Because in $\mathbb{R}^7$ the variety of I_7 consists of one point which is the origin $(0,0,0,0,0,0,0) \in \mathbb{R}^7$, we only need to consider varieties of first six ideals $I_1 - I_6$.

Theorem 4.3. *Suppose that the origin $O:(0,0)$ of system (4.4) is a weak center of a finite order.*
(1) Then the order is at most 3. More precisely, the order is at most 3 (resp. $0,0,3,2,2$) when parameters belong to the variety of the ideal I_1 (resp. $I_2 - I_6$).
(2) Moreover, at most 3 (resp. $0,0,3,2,2$) critical periods can be bifurcated from the weak center O of system (4.4) and there exists a perturbation with exactly 3 (resp. $0,0,3,2,2$) critical periods bifurcated from O when parameters belong to the variety of the ideal I_1 (resp. $I_2 - I_6$).

Proof. When the parameter $\alpha = (a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ belongs to the variety of the ideal I_1, we found that the first four period coefficients of (4.3) are

$$\begin{aligned}
p_{1,2}(\alpha) &= 10a_{02}^2 - a_{02}b_{11} + b_{11}^2 - 3b_{12} - 9b_{30},\\
p_{1,4}(\alpha) &= 1540a_{02}^4 + 700a_{02}^3b_{11} + 21a_{02}^2b_{11}^2 - 2a_{02}b_{11}^3 + b_{11}^4 + 84a_{02}^2b_{12} + 300a_{02}^2b_{30}\\
&\quad +6a_{02}b_{11}b_{12} + 18a_{02}b_{11}b_{30} - 6b_{11}^2b_{12} - 66b_{11}^2b_{30} + 9b_{12}^2 + 54b_{12}b_{30} + 513b_{30}^2,\\
p_{1,6}(\alpha) &= 3403400a_{02}^6 + 3303300a_{02}^5b_{11} + 690690a_{02}^4b_{11}^2 - 699a_{02}^2b_{11}^4 + 281340a_{02}^3b_{11}b_{30}\\
&\quad -139b_{11}^6 + 1261260a_{02}^4b_{12} + 1455300a_{02}^4b_{30} + 346500a_{02}^3b_{11}b_{12} + 11935a_{02}^3b_{11}^3\\
&\quad +7263a_{02}^2b_{11}^2b_{12} - 4995a_{02}^2b_{11}^2b_{30} - 3366a_{02}b_{11}^3b_{12} - 2538a_{02}b_{11}^3b_{30} + 417a_{02}b_{11}^5\\
&\quad +1251b_{11}^4b_{12} + 1377b_{11}^4b_{30} + 6966a_{02}^2b_{12}^2 + 31860a_{02}^2b_{12}b_{30} - 52650a_{02}^2b_{30}^2\\
&\quad +13878a_{02}b_{11}b_{12}b_{30} - 4455a_{02}b_{11}b_{30}^2 - 3321b_{11}^2b_{12}^2 + 2025b_{12}^3 + 5265b_{12}^2b_{30}\\
&\quad -7614b_{11}^2b_{12}b_{30} + 41391b_{11}^2b_{30}^2 - 13365b_{12}b_{30}^2 - 382725b_{30}^3 + 7209a_{02}b_{11}b_{12}^2.
\end{aligned}$$

We omit the expression of $p_{1,8}(\alpha)$, since it is long and the number of its terms is 55.

We compute the decomposition of $\langle p_{1,2}, p_{1,4}, p_{1,6}, p_{1,8}\rangle$ with `minAssGTZ` and obtain $\langle a_{02}, b_{11}^2 - 9b_{30}, b_{12}\rangle$. That is, the condition $p_{1,2} = p_{1,4} = p_{1,6} = p_{1,8} = 0$ yields that $b_{12} = a_{02} = b_{20} = b_{02} = b_{21} = a_{03} = 9b_{30} - b_{11}^2 = 0$, showing that the origin is an isochronous center of system (4.4) in this case by Theorem 2.1.

Solving the equation $p_{1,2}(\alpha) = 0$ we get

$$b_{12} = \tilde{b}_{12} := (10/3)a_{02}^2 - (1/3)a_{02}b_{11} + (1/3)b_{11}^2 - 3b_{30}. \tag{4.5}$$

Substituting (4.5) in $p_{1,4}(\alpha)$, we obtain

$$432b_{30}^2 + 48(a_{02}^2 - b_{11}^2)b_{30} + 1920a_{02}^4 + 672a_{02}^3b_{11} + 48a_{02}^2b_{11}^2 = 0.$$

Thus, when $-1439a_{02}^4 - 504a_{02}^3b_{11} - 38a_{02}^2b_{11}^2 + b_{11}^4 < 0$ the origin O is a weak center of order 1. When $-1439a_{02}^4 - 504a_{02}^3b_{11} - 38a_{02}^2b_{11}^2 + b_{11}^4 \geq 0$, from $p_{1,4}(\alpha) = 0$ we find that

$$b_{30} = \tilde{b}_{30} := \frac{1}{18}\Big(- a_{02}^2 + b_{11}^2 + \sqrt{-1439a_{02}^4 - 504a_{02}^3b_{11} - 38a_{02}^2b_{11}^2 + b_{11}^4}\Big).$$

We now employ the procedure *Reduce* of computer algebra system MATHEMATICA for the set of equalities and inequalities $\{b_{12} = \tilde{b}_{12}, b_{30} = \tilde{b}_{30}, -1439a_{02}^4 - 504a_{02}^3b_{11} - 38a_{02}^2b_{11}^2 + b_{11}^4 \geq$

$0, p_{1,6}(\alpha) = 0, p_{1,8}(\alpha) \neq 0\}$, and find that this semi-algebraic system is fulfilled if and only if $a_{02} \neq 0$ and

$$\begin{aligned}&-128966505300a_{02}^{10} - 131928442900a_{02}^{9}b_{11} - 62892021225a_{02}^{8}b_{11}^{2} - 18497447700a_{02}^{7}b_{11}^{3}\\&-3614043210a_{02}^{6}b_{11}^{4} - 467726370a_{02}^{5}b_{11}^{5} - 37088580a_{02}^{4}b_{11}^{6} - 1472760a_{02}^{3}b_{11}^{7}\\&-19938a_{02}^{2}b_{11}^{8} + 1650a_{02}b_{11}^{9} + 125b_{11}^{10} = 0.\end{aligned} \tag{4.6}$$

Assuming that $a_{02} = 1/2$, we can calculate one of solutions $b_{11} \approx -2.405222225$ from above equation, which indicates the existence of solutions of above equation with respect to parameters a_{02} and b_{11} in real field. Moreover, computing with MATHEMATICA the rank of the matrix

$$\frac{\partial(p_{1,2}, p_{1,4}, p_{1,6})}{\partial(a_{02}, b_{11}, b_{12}, b_{30})},$$

we find that it is equal to 3 when $b_{12} = \tilde{b}_{12}, b_{30} = \tilde{b}_{30}$, $a_{02} \neq 0$ and (4.6) holds. From Theorem 4.1 there exists a perturbation of system (4.4) with exactly 3 critical periods bifurcated from weak center O of order 3 when α belongs to the variety of I_1.

When the parameter $\alpha = (a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ belongs to the variety of the ideal I_2, we have the first period coefficient in (4.3):

$$p_{2,2}(\alpha) \;=\; 10a_{02}^2 - a_{02}b_{11} + b_{11}^2 + 10b_{20}^2,$$

which cannot be equal to zero in the real field, since $10a_{02}^2 - a_{02}b_{11} + b_{11}^2 > 0$ unless $a_{02} = b_{11} = 0$. That is, the center at the origin is of order 0 in this case.

When the parameter $\alpha = (a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ lies in the variety of the ideal I_3, we compute the first period coefficient in (4.3):

$$p_{3,2}(\alpha) \;=\; 10a_{02}^2 - a_{02}b_{11} + b_{11}^2 + 4b_{02}^2 + 10b_{02}b_{20} + 10b_{20}^2,$$

finding that $p_{3,2}(\alpha) \neq 0$ unless all parameters vanish. Thus the center O is of order 0 in this case.

When the parameter $\alpha = (a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ belongs to the variety of the ideal I_4 or I_5, we can see that system (4.4) is a reduced Kukles system. The variety of ideal I_4 (resp. I_5) for center conditions corresponds to the center type K_{III} (resp. K_{II} or K_{IV}) in [46]. Applying Theorems 3.3, 3.4 and 3.7 of [46] we obtain that the order at the origin is at most 3 (resp. 2), and there exists a perturbation with exactly 3 (or 2) critical periods bifurcated from O when parameters belong to the variety of the ideal I_4 (resp. I_5).

When the parameter $\alpha = (a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ belongs to the variety of the ideal I_6, we found that the first three period coefficients in (4.3) are

$$\begin{aligned}p_{6,2}(\alpha) &= 10a_{02}^2 + 10b_{20}^2 - 9b_{30},\\ p_{6,4}(\alpha) &= 1540a_{02}^4 + 200a_{02}^2b_{20}^2 + 1540b_{20}^4 + 300a_{02}^2b_{30} - 3300b_{20}^2b_{30} + 513b_{30}^2,\\ p_{6,6}(\alpha) &= 136136a_{02}^6 + 38808a_{02}^4b_{20}^2 + 13080a_{02}^2b_{20}^4 + 165704b_{20}^6 + 58212a_{02}^4b_{30}\\ &\quad -17496a_{02}^2b_{20}^2b_{30} - 546588b_{20}^4b_{30} - 2106a_{02}^2b_{30}^2 + 341334b_{20}^2b_{30}^2 - 15309b_{30}^3.\end{aligned}$$

Eliminating b_{30} from $p_{6,2}(\alpha) = 0$ we find

$$b_{30} = \hat{b}_{30} := (10a_{02}^2 + 10b_{20}^2)/9.$$

Letting $b_{30} = \hat{b}_{30}$ we obtain from $p_{6,4} = 0$ that

$$47a_{02}^4 - 35a_{02}^2 b_{20}^2 - 28b_{20}^4 = 0,$$

yielding

$$a_{02} = \hat{a}_{02} := \pm\sqrt{35/94 + (3/94)\sqrt{721}}\, b_{20}.$$

Eliminating b_{30} and a_{02} by substituting $b_{30} = \hat{b}_{30}$ and $a_{02} = \hat{a}_{02}$ into $p_{6,6}(\alpha)$, we obtain

$$b_{20}^6(3578681 + 142373\sqrt{721}),$$

which does not vanish if $b_{20} \neq 0$. Therefore, the order of the weak center is at most 2, and there exists a perturbation with exactly 2 critical periods bifurcated from O when parameters belong to the variety of the ideal I_6 by Theorem 4.1, since the rank of the matrix

$$\frac{\partial(p_{6,2}, p_{6,4})}{\partial(a_{02}, b_{20}, b_{30})},$$

is equal to 2 when $b_{30} = \hat{b}_{30}$, $a_{02} = \hat{a}_{02}$ and $b_{20} \neq 0$. Notice that when $b_{20} = 0$ and $a_{02} \neq 0$ the center O is a weak center of order 1, and when $b_{20} = a_{02} = 0$ the center O is either the linear isochronous center or the order is 0. □

5. Conclusion

For cubic generalized Riccati system (1.3), we derived conditions on parameters of the system for the linearizability of the origin, see conditions (1)-(4) of Theorem 2.1.

For the study we have used the approach based on the modular calculations of the set of solutions of polynomial systems, which was used for the first time in [41] and described in details in [42]. The approach can be considered as one between precise symbolic computations and numerical computations since it produces a result which is not completely correct, but correct with high probability – in the sense that it is easily verified if the obtained solutions of a given system of polynomials are correct, but it can happen, that some solutions are lost. Recently an efficient algorithm to verify if the list of solutions obtained with the approach is complete was proposed in [38] however it is not yet implemented in freely available computer algebra systems. The approach can be efficiently applied to study various mathematical models where arises the problem of solving polynomial equations.

When the origin is an isochronous center, we found that system (1.3) has at most three topologically equivalent global structures, which are the global center at the origin, the neighborhood of equilibrium at infinity consists of one elliptic sector and three hyperbolic sectors, and the neighborhood of equilibrium at infinity consists of two hyperbolic sectors and two parabolic sectors, as shown in Theorem 3.1. The last result is the investigation of local bifurcations of critical periods in a neighborhood of the center. We proved that the order of weak center at the origin is at most 3 when parameters belong to the center variety and at most 3 critical periods can be bifurcated from the weak center of system (4.4), as shown in Theorem 4.3.

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Appendix

Here are listed the first two pairs of the linearizability quantities of system (1.3).

$$
\begin{aligned}
i_1 =& 10a_{02}^2 + 9a_{03} + 4b_{02}^2 - a_{02}b_{11} + b_{11}^2 - 3b_{12} + 10b_{02}b_{20} + 10b_{20}^2 - 9b_{30},\\
j_1 =& 2a_{02}b_{02} - b_{02}b_{11} - b_{11}b_{20} + b_{21},\\
i_2 =& 168a_{02}^2a_{03} - 272a_{02}^2b_{02}^2 - 72a_{03}b_{02}^2 - 32b_{02}^4 - 112a_{02}^3b_{11} - 42a_{02}a_{03}b_{11} + 40a_{02}b_{02}^2b_{11} - 21a_{03}b_{11}^2\\
& - 18b_{02}^2b_{11}^2 - 21a_{02}b_{11}^3 - b_{11}^4 + 12a_{02}^2b_{12} - 48b_{02}^2b_{12} + 72a_{02}b_{11}b_{12} - 3b_{11}^2b_{12} + 18b_{12}^2 - 48a_{02}^2b_{02}b_{20}\\
& - 132a_{03}b_{02}b_{20} - 80b_{02}^3b_{20} - 286a_{02}b_{02}b_{11}b_{20} + 47b_{02}b_{11}^2b_{20} - 144b_{02}b_{12}b_{20} - 160a_{02}^2b_{20}^2 - 102a_{03}b_{20}^2\\
& + 12b_{02}^2b_{20}^2 - 306a_{02}b_{11}b_{20}^2 + 61b_{11}^2b_{20}^2 - 114b_{12}b_{20}^2 + 260b_{02}b_{20}^3 + 200b_{20}^4 - 30a_{02}b_{02}b_{21} - 39b_{02}b_{11}b_{21}\\
& + 84a_{02}b_{20}b_{21} - 96b_{11}b_{20}b_{21} + 27b_{21}^2 + 132a_{02}^2b_{30} + 81a_{03}b_{30} - 66b_{02}^2b_{30} + 207a_{02}b_{11}b_{30} - 6b_{11}^2b_{30}\\
& - 6a_{02}^2b_{11}^2 + 81b_{12}b_{30} - 498b_{02}b_{20}b_{30} - 498b_{20}^2b_{30} + 135b_{30}^2,\\
j_2 =& 224a_{02}^3b_{02} + 240a_{02}a_{03}b_{02} - 16a_{02}b_{02}^3 - 184a_{02}^2b_{02}b_{11} + 6a_{03}b_{02}b_{11} + 40b_{02}^3b_{11} + 124a_{02}b_{02}b_{11}^2 - 11b_{02}b_{11}^3\\
& - 156a_{02}b_{02}b_{12} + 54b_{02}b_{11}b_{12} - 120a_{02}^3b_{20} - 108a_{02}a_{03}b_{20} + 104a_{02}b_{02}^2b_{20} + 40a_{02}^2b_{11}b_{20} + 27a_{03}b_{11}b_{20}\\
& + 38b_{02}^2b_{11}b_{20} + 77a_{02}b_{11}^2b_{20} - 8b_{11}^3b_{20} + 24a_{02}b_{12}b_{20} + 39b_{11}b_{12}b_{20} + 140a_{02}b_{02}b_{20}^2 - 64b_{02}b_{11}b_{20}^2\\
& - 120a_{02}b_{20}^3 - 50b_{11}b_{20}^3 - 48a_{02}^2b_{21} - 45a_{03}b_{21} - 42b_{02}^2b_{21} - 87a_{02}b_{11}b_{21} - 27b_{12}b_{21} - 6b_{02}b_{20}b_{21}\\
& + 6b_{11}^2b_{21} + 30b_{20}^2b_{21} - 270a_{02}b_{02}b_{30} + 105b_{02}b_{11}b_{30} + 108a_{02}b_{20}b_{30} + 84b_{11}b_{20}b_{30} - 36b_{21}b_{30}.
\end{aligned}
$$

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